Int. J. Heat Mass Transfer. Vol. 36, No. 4, pp. 1043–1048, 1993
Printed in Great Britain

0017-9310/93 $6.00+0.00
Pergamon Press Ltd

A problem of measurement of temperature solved using the one-dimensional heat equation

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(*Received* 2 *February* 1992)

Abstract—A numerical method to determine the variations of temperature is presented. The problem of measurement of the variations of temperature is mathematically modelled with the help of the one-dimensional heat conduction equation. This equation describes the heat transfer through an insulated long cylindrical rod. The variation of the temperature which is produced near one end of the rod is determined using some measurements of temperature in the other end of the rod and the mathematical model. This rod works as an attenuator of high temperatures and as an amplifier of low temperatures.

INTRODUCTION

IT IS KNOWN that for the measurement of temperature we use some temperature transducers. They are divided in two classes: passive transducers and self-generating transducers. Thermistors, resistance–temperature detectors, semiconductor temperature sensors are in the first class. Thermocouples are important elements in the second class [1, 2].

In this work the authors consider the problem of measurement of the temperature in which the unknown temperature is not determined directly using temperature transducers. It is determined through the formulation of the measurement problem as an optimal control problem. The last problem can be solved using a numerical computer.

We use an ensemble which has as the principal component a long cylindrical rod with a diameter that is as small as possible. The material from which the rod is developed is chosen so that it satisfies the following requirements:

- The melting temperature must be higher than the temperature to be measured;
- Perfectly homogeneous;
- The thermal conductivity coefficient and the thermal exchange coefficient must vary as little as possible with the temperature.

Figure 1 shows the scheme of the ensemble.

The rod will be covered by a thermic insulating layer made from ceramics fibre. If necessary, above it we shall place a protected cover against corrosive action of the medium in which the measurements are realized.

One end of the rod will not be covered by the insulated layer. This will be the end which will be in contact with the point in which we want to measure the variation of the temperature. This end we shall name 'the hot end'.

The other end we shall name 'the cold end'. In a small area of this end we place equidistantly, on a generating line of the rod, some temperature transducers, e.g. thermocouples. These shall be placed directly on the rod, under the thermic insulating layer.

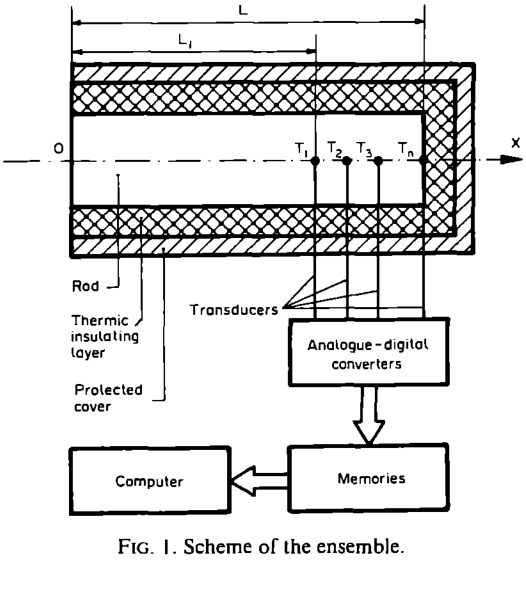

FIG. 1. Scheme of the ensemble.

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NOMENCLATURE

C one positive real constant

C_1 thermal exchange coefficient between rod and medium for measurement [m^{-1}]

$C^{\kappa}(\Omega)$, ($\kappa \in N$) set of the continuous differentiable functions definite on Ω till the order κ

$C([0, T]; X)$ space of the continuous functions from $[0, T]$ to X

$H^S(\Omega) = W^{S,2}(\Omega)$, ($S \in N$) space of the functions $Y \in L^2(\Omega)$ which have all the derivatives till the order S belonged at $L^2(\Omega)$

Inf greatest lower bound

L length of the rod [mm]

L_1 length of the part of the rod on which the temperature transducers are not placed [mm]

$L^P(\Omega)$, ($P \in N^*$) space of the functions P-sumable on Ω

$L^P(0, T; X)$, (X-Banach space over R) space of the functions P-sumable from $[0, T]$ to X

n number of the temperature transducers on the rod

N natural numbers system

N^* non-zero natural numbers system

R real numbers system

R_+^* positive and non-zero real numbers system

R^n linear space of n dimension on the real corp

t time [s]

T time of measurement of the variation of temperature which is produced near the hot end of the rod [s]

U unknown temperature which must be measured [K]

$U_n \xrightarrow{W} U$ convergence of the sequence U_n at U in the topological space W

$\mathbf{U} = [M_1, M_2]$ temperature range where the unknown temperature U could be situated [K]

$W^{K,P}([0, T]; X)$, (X-Banach space) space of the functions $Y \in L^P(0, T; X)$ so that $\partial^j Y/\partial t^j \in L^P(0, T; X)$, $1 \leqslant j \leqslant K$, $K \in N$

x distance on the rod against the hot end [mm]

$Y(t, x)$ temperature of the rod at the moment t in the point x [K]

$Y_0(x)$ initial temperature in the point x of the rod, $Y_0(x) = Y(0, x)$ [K]

$\hat{Y}(t, x)$ temperature of a point from the cold end of the rod which is measured by the temperature transducers [K]

Y', $Y'(t)$, Y_t, $Y_t(t)$, $Y_t(t, x)$ derivative of the following function Y: $(0, T) \times (0, L) \to X$ of t

Y_x, $Y_x(t, x)$ derivative of the following function Y: $(0, T) \times (0, L) \to X$ of x

Y_{xx}, $Y_{xx}(t, x)$ derivative of the following function Y_x: $(0, T) \times (0, L) \to X$ of x.

Greek symbols

α thermal conductivity coefficient of the rod [$m^2 s^{-1}$]

$\partial/\partial\nu$ exterior normal derivative at $\partial\Omega$

$\partial\Omega$ boundary of Ω

Ω an open and bounded subset of R^n.

We consider there are n transducers on the rod. They are coupled with a computer using some analogue–digital converters and some memories.

We shall explain the work of this ensemble when the unknown temperature is higher than the temperature of the medium.

We wish to measure the unknown variation of the temperature which appears near the hot end of the rod. We consider this temperature U is constant during the time of measurement T. This variation of the temperature is as a step function.

The length of the rod L will be determined by the practical situation.

During time T, the variations of the temperature are measured with the n transducers from the cold end.

It is clear from the above considerations that the temperature of the cold end will be much smaller than the measured temperature.

Thus, it can be considered that this ensemble realizes the function of an attenuation of temperature. The central idea of measurement is the calculation of the unknown temperature using the variations of temperature of those n transducers in the time of measurement T.

Certainly the ensemble works in a similar manner when we want to measure temperatures smaller than the temperature of the medium. In this case it can be considered that it realizes the function of an amplifier of temperature.

THEORETICAL APPROACH

The phenomenon described above is modelled by a partial differential equation of parabolic type, named the heat equation,

$$Y_t(t, x) - \alpha Y_{xx}(t, x) = 0, \quad t \in (0, T), \quad x \in (0, L) \tag{1.1}$$

with the boundary conditions, of the Neumann type:

$$\frac{\partial Y}{\partial v}(t,0) = -Y_x(t,0) = C_1(U - Y(t,0)), \quad t \in (0,T)$$

$$\frac{\partial Y}{\partial v}(t,L) = Y_x(t,L) = 0, \quad t \in (0,T) \qquad (1.2)$$

and the initial condition:

$$Y(0,x) = Y_0(x), \quad x \in (0,L) \qquad (1.3)$$

where $\alpha \in R_+^*$, $C_1 \in R_+^*$, $Y_0 \in H^1(0,L)$.

The value of the measured temperature U will be determined from the solution of the following problem.

Minimize:

$$\Phi(U) = \int_0^T \int_{L_1}^L |Y(t,x) - \hat{Y}(t,x)|^2 \, dx \, dt \qquad (P)$$

where $\hat{Y} \in L^2((0,T) \times (L_1, L))$, $U \in \mathbf{U} = [M_1, M_2] \subset R$, and $Y(t,x)$ is the solution of the above problem (1.1)–(1.3), solution which corresponds to U. In the above relation, $\hat{Y}(t,x)$ represents the temperatures of the points from the cold end of the rod. They are obtained from an interpolation technique of the experimental data acquired by the transducers $T_1, T_2, \ldots, T_n$.

The functional $\Phi(U)$ has the semnification of the quadratic mean error between the values which were calculated and those which were measured in the cold end.

It is observed that the problem (1.1)–(1.3) admits only a solution $Y \in W^{1,2}([0,T]: L^2(0,L)) \cap L^2(0,T; H^1(0,L))$. In addition, the function $Y \to Y(t)$ is weak continuous from $[0,T]$ to $H^1(0,T)$, that is if $U_n \xrightarrow{u} U$ then $\langle U_n, v\rangle \to \langle U, v\rangle$, $\forall v \in H^1(0,L)$ [3].

We shall observe that the problem (P) admits at least an optimal solution, that is, it exists (U^*, v^*) so that $U^* \in \mathbf{U} = [M_1, M_2]$ and Y^* is the optimal solution of the problem (1.1)–(1.3) which corresponds to U^*.

That is equivalent to:

$$\Phi(U^*) = \int_0^T \int_{L_1}^L |Y^*(t,x) - \hat{Y}(t,x)|^2 \, dx \, dt = \inf_{U \in \mathbf{U}} \Phi(U).$$

For proving, we note $d = \inf_{U \in \mathbf{U}} \Phi(U)$. It is evident that $d \geq 0$. From this it results that it exists $\{(U_n, Y_n)\}_{n \in N}$ so that Y_n is the solution of the problem (1.1)–(1.3) which corresponds to $U \in \mathbf{U}$ and

$$d \leq \int_0^T \int_{L_1}^L |Y_n(t,x) - \hat{Y}(t,x)|^2 \, dx \, dt < d + \frac{1}{n}. \qquad (2)$$

Because $\{U_n\}_{n \in N}$ is a bounded system of functions, it means that a subsystem of this (for which we use identical notation), one obtains: $U_n \to U_n^*$ (the Cesàro lemma).

Multiplying (1.1) with Y_n and Y_n', respectively, we obtain:

$$\|Y_n(t)\|_{L^2(0,L)} + \|Y_n'\|_{L^2((0,T)\times(0,L))} \leq C,$$

where C is a positive real constant. The proof of this result is clear. For example if we multiply (1.1) by Y_n, we obtain:

$$Y_n Y_{n_t} - \alpha Y_n Y_{n_{xx}} = 0.$$

which is equivalent to

$$\tfrac{1}{2}(Y_n)_t^2 - \alpha Y_{n_{xx}} Y_n = 0.$$

Integrating on $(0,T) \times (0,L)$ we obtain:

$$\frac{1}{2}\int_0^T \int_0^L (Y_n)_t^2 \, dx \, dt - \alpha \int_0^T \int_0^L Y_{n_{xx}} Y_n \, dx \, dt = 0,$$

and if we use Green's theorem and relation (1.2):

$$\frac{1}{2}\int_0^L Y_n^2 \Big|_0^t \, dx$$

$$+ \alpha \int_0^T \left(\int_0^L \nabla Y_n^2 \, dx - \int_{\Gamma[0,L]} Y \frac{\partial Y}{\partial v} \partial\sigma \right) dt = 0$$

$$\Leftrightarrow \frac{1}{2}\int_0^L Y_n^2(t,x) \, dx - \frac{1}{2}\int_0^L Y_n^2(0,x) \, dx$$

$$+ \alpha \int_0^L \int_0^T \nabla Y_n^2 \, dx \, dt = 0$$

$$\Leftrightarrow \int_0^L Y_n^2(t,x) \, dx = \int_0^L Y_n^2(0,x) \, dx$$

$$- 2\alpha \int_0^T \int_0^L \nabla Y_n^2 \, dx \, dt \Rightarrow \|Y_n(t)\|_{L^2(0,L)} \leq C.$$

In the same manner it results that:

$$\|Y_n'\|_{L^2((0,T)\times(0,L))} \leq C.$$

Thus, we obtained the above inequality.

Using the Arzelà lemma it results that the system of functions Y_n is relatively compact in $C([0,T]; L^2(0,L))$. From that for a subsystem of functions of Y_n (using identical notation) one obtains:

$$Y_n \to Y_n^* \quad \text{in } C([0,T]; L^2(0,L)) \qquad (3)$$

and that Y^* is the solution of the problem (1.1)–(1.3), the solution which corresponds to U^*.

From (2) and (3) it follows therefore that $\Phi(U_n) \to \Phi(U^*)$.

Hence, (U^*, Y^*) is an optimal pair for the problem (P).

In succession, we shall find the optimal conditions for the problem (P).

Theorem. The pair (U^*, Y^*) is an optimal one for the problem (P) if and only if $P \in W^{1,2}([0,T]; L^2(0,L)) \cap L^2(0,T; W^{2,2}(0,L))$ so that P is a solution of the problem:

$$P_t+\alpha P_{xx} = \begin{cases} Y^*-\hat{Y}, & (t,x)\in(0,T)\times(L_1,L) \\ 0, & (t,x)\in(0,T)\times(0,L_1) \end{cases} \quad (4.1)$$

$$\frac{\partial P}{\partial v}(t,0) = -P_x(t,0) = -C_1 P, \quad t\in(0,T)$$

$$\frac{\partial P}{\partial v}(t,L) = P_x(t,L) = 0, \quad t\in(0,T) \quad (4.2)$$

$$P(T,x) = 0, \quad x\in(0,L) \quad (4.3)$$

and it happens:

$$U^* = \begin{cases} M_2, & \text{if } \int_0^T P(t,0)\,dt > 0 \\ M_1, & \text{if } \int_0^T P(t,0)\,dt < 0. \end{cases} \quad (5)$$

Remark. Because $P\in L^2(0,T;W^{2,2}(0,L))$ and $W^{2,2}(0,L)\subset C((0,L))$ therefore it is significant to speak about

$$\int_0^T P(t,0)\,dt.$$

Proof: We consider $P\in W^{1,2}([0,T];\ L^2(0,L))\cap L^2(0,T;W^{2,2}(0,L))$ the solution of the problem (4.1)–(4.3).

The existence of the solution of the problem (4.1)–(4.3) is demonstrated in more general conditions in ref. [4].

If (U^*, Y^*) is an optimal pair for the problem (P), then:

$$\int_0^T\int_{L_1}^L |Y^*(t,x)-\hat{Y}(t,x)|^2\,dx\,dt \leqslant \int_0^T\int_{L_1}^L |Y^{U^*+\lambda v}(t,x)-\hat{Y}(t,x)|^2\,dx\,dt \quad (6)$$

$\forall \lambda>0,\quad \forall v\in R,\quad$ if $U^*\in(M_1,M_2)$,
$\forall v>0,\quad$ if $U^*=M_1$, and
$\forall v<0,\quad$ if $U^*=M_2$

where $Y^{U^*+\lambda v}$ is the solution of the problem (1.1)–(1.3) which corresponds to $U^*+\lambda v$.

From (6) it results that:

$$\int_0^T\int_{L_1}^L (Y^{U^*+\lambda v}-Y^*)(Y^{U^*+\lambda v}+Y^*-2\hat{Y})\,dx\,dt \geqslant 0.$$

We divide the above expression through λ and then we realize $\lambda\to 0$. We obtain:

$$\int_0^T\int_{L_1}^L Z(t,x)[Y^*(t,x)-\hat{Y}(t,x)]\,dx\,dt \geqslant 0, \quad (7)$$

where $Z\in W^{1,2}([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$ is the solution of the problem:

$$Z_t-\alpha Z_{xx} = 0, \quad (t,x)\in((0,T)\times(0,L)) \quad (8.1)$$

$$\frac{\partial Z}{\partial v}(t,0) = -Z_x(t,0) = -C_1(Z(t,0)-v), \quad t\in(0,T)$$

$$\frac{\partial Z}{\partial v}(t,L) = Z_x(t,L) = 0, \quad t\in(0,T) \quad (8.2)$$

$$Z(0,x) = 0, \quad x\in(0,L). \quad (8.3)$$

We multiply (4.1) by Z, integrate on $(0,T)\times(0,L)$ and obtain:

$$\int_0^T\int_0^L (P_tZ+\alpha P_{xx}Z)\,dx\,dt = \int_0^T\int_{L_1}^L Z(Y^*-\hat{Y})\,dx\,dt$$

$$\Leftrightarrow \int_0^T\int_0^L (P_tZ-\alpha P_xZ_x)\,dx\,dt + \alpha\int_0^T\left[\frac{\partial P}{\partial v}(t,0)Z(t,0)+\frac{\partial P}{\partial v}(t,L)Z(t,L)\right]dt = \int_0^T\int_{L_1}^L Z(Y^*-\hat{Y})\,dx\,dt.$$

Using (4.2), (4.3), (8.3) we get:

$$-\int_0^T\int_0^L (PZ_t+\alpha P_xZ_x)\,dx\,dt - \alpha\int_0^T C_1P(t,0)Z(t,0)\,dt = \int_0^T\int_{L_1}^L Z(Y^*-\hat{Y})\,dx\,dt. \quad (9)$$

We multiply (8.1) by P, integrate on $(0,T)\times(0,L)$ and acquire:

$$\int_0^T\int_0^L (Z_tP-\alpha Z_{xx}P)\,dx\,dt = 0$$

$$\Leftrightarrow \int_0^T\int_0^L (Z_tP+\alpha Z_xP_x)\,dx\,dt - \alpha\int_0^T\left[\frac{\partial Z}{\partial v}(t,0)P(t,0)+\frac{\partial Z}{\partial v}(t,L)P(t,L)\right]dt = 0.$$

Using (8.2) we obtain:

$$\int_0^T\int_0^L (Z_tP+\alpha Z_xP_x)\,dx\,dt = -\alpha\int_0^T C_1(Z(t,0)-v)P(t,0)\,dt$$

$$\Leftrightarrow \int_0^T\int_0^L (Z_tP+\alpha Z_xP_x)\,dx\,dt = -\alpha\int_0^T C_1Z(t,0)P(t,0)\,dt + \alpha\int_0^T C_1vP(t,0)\,dt. \quad (10)$$

From (9) and (10) we get:

$$\int_0^T \int_{L_1}^{L} Z(Y^* - \hat{Y})\,\mathrm{d}x\,\mathrm{d}t = -\alpha C_1 \int_0^T P(t,0)v\,\mathrm{d}t. \tag{11}$$

From (7) and (11) we get:

$$-\alpha C_1 v \int_0^T P(t,0)\,\mathrm{d}t \geqslant 0.$$

Hence,

$$\int_0^T P(t,0)\,\mathrm{d}t > 0 \Rightarrow v \leqslant 0 \Rightarrow U^* = M_2$$

$$\int_0^T P(t,0)\,\mathrm{d}t < 0 \Rightarrow v \geqslant 0 \Rightarrow U^* = M_1.$$

Consequently, (5) is true, hence the necessary condition was demonstrated.

The sufficiency of the conditions (4) and (5) results immediately from the linearity of the problem (1.1)–(1.3) and the convexity of the cost functional $\Phi(U)$ [5].

Remark. The problem (P) can be looked at as a problem with impulsional control on the boundary, for the heat equation, in the domain $\Omega = (0, L) \subset R^N$, with $N = 1$. In the case $N = 2$ or $N = 3$ the problem is very difficult [6]. It could certainly use a model in which $\Omega \subset R^2$ or $\Omega \subset R^3$. We stopped here with the model which appeared to be suitable to the enunciated problem.

NUMERICAL APPROACH

For the numerical calculation of the temperature U, which is an optimal control problem, as is seen before, we shall use one type of gradient algorithm.

In view of the relations (1.1)–(1.3), (4.1)–(4.3), (5), we consider the following iterative algorithm:

$$U^{j+1} = \delta_j U^j + (1-\delta_j)v^j,$$

where:

$$v^j = \begin{cases} M_2, & \text{if } \int_0^T P^j(t,0)\,\mathrm{d}t > 0 \\ M_1, & \text{if } \int_0^T P^j(t,0)\,\mathrm{d}t < 0. \end{cases}$$

U^j, P^j are computed from:

$$\begin{cases} Y_t^j - \alpha Y_{xx}^j = 0, \quad (t,x) \in (0,T) \times (0,L) \\ \dfrac{\partial Y^j}{\partial v}(t,0) = C_1(Y^j - U^j) \\ \qquad\qquad\qquad\qquad , \quad t \in (0,T) \\ \dfrac{\partial Y^j}{\partial v}(t,L) = 0, \\ Y^j(0,x) = Y_0(x), \quad x \in [0,L] \end{cases}$$

$$\begin{cases} P_t^j + \alpha P_{xx}^j = \begin{cases} Y^j - \hat{Y}, & (t,x) \in (0,T) \times (L_1, L) \\ 0, & (t,x) \in (0,T) \times (0, L_1] \end{cases} \\ \dfrac{\partial P^j}{\partial v}(t,0) = -C_1 P^j \\ \qquad\qquad\qquad\qquad , \quad t \in (0,T) \\ \dfrac{\partial P^j}{\partial v}(t,L) = 0 \\ P^j(T,x) = 0, \quad x \in [0,L] \end{cases}$$

and $\delta_j \in [0, 1]$ are chosen so that:

$$\Phi(U^{j+1}) \leqslant \Phi(U), \quad \forall\, U = \mu U^j + (1-\mu)v^j, \quad \mu \in [0,1]$$

$$\Phi(U) = \int_0^T \int_{L_1}^{L} |Y(t,x) - \hat{Y}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

These kind of techniques have been investigated in refs. [3, 7, 8].

CONCLUSIONS

Generally, temperature transducers work in the stationary state. In our problem the above ensemble can be considered as a temperature transducer which works in the transient state. It observes that this ensemble utilizes a small part of the beginning of the transient state of the heat transfer in the rod.

Remark. Because the time T chosen is rather small, the requirements that the thermal conductivity coefficient and the thermal exchange coefficient must vary as little as possible with the temperature occur naturally. Nevertheless it is observed that in real situations the thermal exchange coefficient varies significantly with the temperature.

From this point of view the authors show an improvement of the algorithm above.

We design an instrument for measuring the temperature U from the range $[U_{\min}, U_{\max}]$.

We shall determine experimentally the dependence $C_1(U)$, where $U \in [U_{\min}, U_{\max}]$. These values will be memorized by a computer. We shall present in a future paper the details about this experimental determination.

It is considered that the unknown temperature U is situated in the range $[U_1, U_2] \subset [U_{\min}, U_{\max}]$.

The range $[U_1, U_2]$ is divided in some subranges and for each of them we consider that C_1 has a constant value. For each of these subranges we compute the value of U with the above type gradient algorithm and the value of functional $\Phi(U)$.

We shall keep the subrange of the temperature where we obtained the smallest value for Φ. We note this subrange with $[U_1, U_2]$ and shall continue as above.

In the practical problems we can consider the following stop criterions:

- The length of the range $[U_1, U_2]$;
- The insignificant diminuation of the values of the functional Φ;

• The insignificant variation of the values of U.

In a future paper we shall present some experimental results.

REFERENCES

1. A. D. Helfrick and W. D. Cooper, *Modern Electronic Instrumentation and Measurement Techniques*, Chap. 11.5. Prentice Hall, U.S.A. (1991).
2. L. D. Jones and A. F. Chin, *Electronic Instruments and Measurements*, pp. 290–298. Prentice Hall, U.S.A. (1991).
3. V. Arnautu and V. Barbu, Optimal control of free boundary in a two-phase Stefan problem, *Preprint Series in Mathematics*, No. 11. INCREST Bucharest, Romania (1985).
4. H. Brézis, Problèmes unilatéraux, *J. Math. pures appl.* **51**, 1–168 (1972).
5. V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces*, pp. 357–371. Reidel, Dordrecht, Boston (1986).
6. S. Anita, Optimal control of parabolic problems with boundary impulse-controllers, *Ann. Univ.* Iasi, **38**, Romania (1992).
7. V. Thomée, Finite difference methods for linear parabolic equations, *Handbook of Numerical Analysis* (Edited by P. G. Ciarlet and J. L. Lions), Vol. 1, pp. 9–196. North-Holland (1990).
8. P. A. Raviart and J. M. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson, Paris (1983).